DIFFERENTIAL EQUATIONS

Generally an equation which Contains derivative of dependent variables relative to independent variable.

Order of a diff^n eq^n

highest derivative of dependent variables relative to independent variable

Degree of diff^n eq^n

Power of Heighest derivative after removing the radical sign and fraction.
main point to study Power of diff^n eq^n that diff eq^n must be polynomial in y', y'', y'''.... etc

Examples Find order and power of given diff eq^n

(1) $\left(\frac{d^2y}{dx^2}\right)^2 + \frac{dy}{dx} = 0 \rightarrow$ order = 2, Power = 2

(2) $\left(\frac{d^2y}{dx^2}\right)^3 + \cos\left(\frac{dy}{dx}\right) = 0 \rightarrow$ order = 2, Power = not defined

(3) $\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2} = \frac{d^2y}{dx^2} \xrightarrow{\text{removing Fraction}} \left[1+\left(\frac{dy}{dx}\right)^2\right]^3 = \left(\frac{d^2y}{dx^2}\right)^2$

order = 2, Power = 2

Methods to solve differential eq^n

① eq^n Solvable by Separation of Variable

this method is useful when diff. eq^n is in general form Like $\boxed{f_1(x)\,dx + f_2(y)\,dy = 0}$

then sol^n Can be obtained easily

Ex: ① $\frac{dy}{dx} = \frac{1+y^2}{1+x^2} \Rightarrow \int \frac{dy}{1+y^2} = \int \frac{dx}{1+x^2}$

$$\boxed{\tan^{-1}y = \tan^{-1}x + C}$$

ex ② $\sec^2 x \tan y \, dx + \sec^2 y \tan x \, dy = 0$

Soln Separate the variables

$$\frac{\sec^2 x}{\tan x} dx + \frac{\sec^2 y}{\tan y} dy = 0$$

$$\therefore \int \frac{\sec^2 x}{\tan x} dx + \int \frac{\sec^2 y}{\tan y} dy = 0$$

$$\log \tan x + \log \tan y = \log C$$

$$\boxed{\tan x \tan y = C}$$

$$\left[\log m + \log N = \log MN \right]$$

Ex. ③ $\frac{dy}{dx} - x \tan(y-x) = 1$

imp

Soln its not looking like eqn solvable by separation but it is

Let $y - x = v \Rightarrow \frac{dy}{dx} - 1 = \frac{dv}{dx}$

Put the values

$$\frac{dv}{dx} + 1 - x \tan v = 1$$

$$\frac{dv}{dx} = x \tan v \Rightarrow \int \frac{1}{\tan v} dv = \int x \, dx$$

$$\log|\sin v| = x^2/2 + C$$

or $$\log|\sin(y-x)| = x^2/2 + C$$

② Homogeneous diff eqn

Form $\frac{dy}{dx} = \frac{f_1(x,y)}{f_2(x,y)}$ where $f_1(x,y)$ and $f_2(x,y)$ are homogeneous func of same degree ex. 2 $x \sin\left(\frac{y}{x}\right) dy = \left\{ y \sin\left(\frac{y}{x}\right) - x \right\} dx$

Solⁿ

$$\frac{dy}{dx} = \frac{(y/x)\sin(y/x) - 1}{\sin(y/x)}$$

Put $y = vx \Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$

$$v + x\frac{dv}{dx} = \frac{v\sin v - 1}{\sin v}$$

$$x\frac{dv}{dx} = \frac{-1}{\sin v} \Rightarrow -\int \sin v \, dv = \int \frac{dx}{x}$$

$$\boxed{\cos v + \log c = \log x}$$

3 Eqⁿ Reducible to a Homogeneous eqⁿ :

gen. form $\boxed{\frac{dy}{dx} = \frac{ax + by + c_1}{Ax + By + c_2}}$ here a, b, A, B, c_1 and c_2 are Constant

This eqⁿ Can be reduced into Homogeneous

Condition-1 when $a/A \neq b/B$

then x and y are interchanged by X and Y which are related as

$$x = X + h, \quad y = Y + k$$

where h and k are arbitrary Constant, choosen in such a way that D.E Reduced to Homogeneous

so

$$ah + bk + c_1 = 0$$

$$Ah + Bk + c_2 = 0$$

from these two eqⁿ h and k Can be obtained. So New form $\boxed{\frac{dY}{dX} = \frac{aX + bY}{AX + BY}}$

To solve homogeneous D.E

Put $y = vx$

Reduce D.E in v and x variable form such reduced D.E can be solved by variable separation method.

Ex ① $(x^2y - 2xy^2)dx - (x^3 - 3x^2y)\,dy = 0$

Sol^n given D.E is homogeneous (How=?)

if you dont know that how we can say that eq^n is homogeneous or not then it is very simple.

important

Just add the power of ~~each term~~ all variables in each term. if each term have same degree then eq^n is homogeneous otherwise not.

look above eq^n First term is x^2y^1 so $2+1=3$

Second term is $-2x^1y^2$ so $1+2=3$

and so on

Now comes on the point

$$(x^2y - 2xy^2)\,dx - (x^3 - 3x^2y)\,dy = 0$$

$$\frac{dy}{dx} = \frac{x^2y - 2xy^2}{x^3 - 3x^2y}$$

Put $y = vx \Rightarrow dy/dx = v + x\frac{dv}{dx}$

$$v + x\frac{dv}{dx} = \frac{v - 2v^2}{1 - 3v} \Rightarrow x\frac{dv}{dx} = \frac{v^2}{1 - 3v}$$

$$\int \frac{(1-3v)}{v^2}\,dv = \int \frac{dx}{x}$$

Now this is eq^n solvable by separation of variable

Ans $\Rightarrow \boxed{cy^3 = x^2 e^{-x/y}}$

4. Linear differential eqⁿ

Gen. form of first order L.D.E

$$\frac{dy}{dx} + P(x)\, y = Q(x) \qquad \boxed{I.F = e^{\int P(x)\,dx}}$$

integration factor

Solⁿ $\boxed{y\,(I.F) = \int (I.F)\, Q(x)\, dx}$

ex. ① $\frac{dy}{dx} + y \tan x = \sec x$

Solⁿ Compare with General form

$P(x) = \tan x \quad , \quad Q(x) = \sec x$

$I.F = e^{\int \tan x\, dx} = e^{\log|\sec x|} = \sec x$

So

$y . \sec x = \int \sec x . \sec x\, dx$

$y \sec x = \tan x + c$

$\boxed{y = \sin x + c \cos x}$

Ex. ② $(1+y^2)\,dx = (\tan^{-1} y - x)\,dy$

$$\frac{dx}{dy} + \left(\frac{1}{(1+y^2)}\right) x = \frac{\tan^{-1} y}{(1+y^2)}$$

Solⁿ here y is independent variable

so Gen L.D.E Form $\frac{dx}{dy} + P(y)\, x = Q(y)$

Now $I.F = e^{\int P(y)\, dy} = e^{\tan^{-1} y}$

$$x\cdot(I.F) = \int I.F \; Q(y)\, dy + c$$

$$x e^{\tan^{-1}y} = \int e^{\tan^{-1}y} \left(\frac{\tan^{-1}y}{1+y^2}\right) dy + C$$

Let $\tan^{-1}y = t \Rightarrow \frac{1}{1+y^2}\, dy = dt$

$$x e^{t} = \int e^{t} t \, dt + C$$

$$x e^{t} = t e^{t} - e^{t} + C$$

$$x = t - 1 + C e^{-t}$$

$$\boxed{x = \tan^{-1}y - 1 + C e^{-\tan^{-1}y}}$$

5. Reducible to L.D.E

Gen. form $\boxed{\frac{dy}{dx} + P(x)\, y = Q(x)\, y^n}$

divide the eqn by y^n

$$y^{-n}\frac{dy}{dx} + P(x)\, y^{1-n} = Q(x)$$

let $y^{1-n} = v$, $\quad n \neq 1$

then $(1-n)\, y^{-n}\frac{dy}{dx} = \frac{dv}{dx}$

New form $\frac{1}{(1-n)} y^{-n} \frac{dv}{dx} + P(x)\, v = Q(x)$

or

$$\boxed{\frac{dv}{dx} + P(1-n)\, v = Q(1-n)}$$

this is L.D.E.

Ex. 1 $\quad \frac{dy}{dx} + xy = x^3y^3$

Sol^n here y^n is y^3 so first divide the eq^n by y^3

$$y^{-3}\frac{dy}{dx} + xy^{1-3} = x^3 \quad \text{—— (1)}$$

Now let $\quad y^{1-3} = y^{-2} = v$

$$\Rightarrow -2y^{-3}\frac{dy}{dx} = \frac{dv}{dx}$$

Put these values in eq (1)

$$-\frac{1}{2}\cdot\frac{dv}{dx} + xv = x^3$$

$$\boxed{\frac{dv}{dx} - 2xv = -2x^3}$$

this is L.D.E can be solved by previous meth
here $P(x) = -2x$
$Q(x) = -2x^3$

6. Exact diff. eq^n

$$\boxed{M\,dx + N\,dy = 0}$$

(here M and N are functions of x, y)

necessary and sufficient condition for above eq^n to be exact is

$$\boxed{\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}}$$

then sol^n of Exact d.E in three steps

(i) $U(x, y) = \int M\,dx$ (Integrate M respect to x, keeping y constant)

(ii) then find $\frac{\partial U}{\partial y}$ ← Partial derivative

(iii) $V(y) = \int \left(N - \frac{\partial U}{\partial y}\right) dy$

Desired Sol^n

$$\boxed{U(x, y) + V(y) = C}$$

Ex: $(x+y)^2 dx - (y^2 - 2xy - x^2)\,dy = 0$

Sol^n Compare eq^n by $M\,dx + N\,dy = 0$

$$M = (x+y)^2 \Rightarrow \frac{\partial M}{\partial y} = 2(x+y)$$

$$N = -(y^2 - 2xy - x^2) \Rightarrow \frac{\partial N}{\partial x} = -(2y - 2\cancel{x}) = 2(x+y)$$

$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ hence given D.E is exact.

$\therefore$ (i) $u(x,y) = \int M\,dx = \int (x+y)^2 dx = \frac{(x+y)^3}{3}$ (keeping y constant)

(ii) $\frac{\partial u}{\partial y} = (x+y)^2$

(iii) $N - \frac{\partial u}{\partial y} = -y^2 + 2xy + x^2 - (x+y)^2 = -2y^2$

(iv) $v(y) = \int \left(N - \frac{\partial u}{\partial y}\right) dy = -\frac{2y^3}{3}$

Sol^n of given D.E

$$u(x,y) + v(y) = c_1$$

$$\frac{(x+y)^3}{3} - \frac{2y^3}{3} = c_1$$

$$\boxed{(x+y)^3 - 2y^3 = c}$$ where $c = 3c_1$ New Constant

7. Reducible to Exact D.E:.

I) Integrating Factor of the Homogeneous eq^n

if $M\,dx + N\,dy = 0$ is a homogeneous eq^n then Integration factor (I.F) $= \frac{1}{Mx + Ny}$

where $Mx + Ny \neq 0$

NOTE when $Mx + Ny = 0$ then $\frac{M}{N} = -y/x$

So eq^n will be

$$\frac{M}{N} dx + dy = 0$$

$$-\frac{y}{x} dx + dy = 0$$

Now variable Separation method is sufficient to solve this

EX: $(x^2y - 2xy^2)\,dx - (x^3 - 3x^2y)\,dy = 0$

Sol^n You know given eq^n is homogeneous

so $$I.F. = \frac{1}{Mx+Ny} = \frac{1}{(x^2y - 2xy^2)x - (x^3 - 3x^2y)y}$$

$$\boxed{I.F = \frac{1}{x^2y^2}}$$

Now muliply the given D.E with I.F, obtained eq^n will be exact D.E which Can be solved by previous method.

(II) I.F of $f_1(xy)\, y\, dx + f_2(xy)\, x\, dy = 0$

find $I.F = \frac{1}{Mx - Ny}$

then multiply given D.E with I.F, obtained eq^n will be exact

(III) when $\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = f(x)$ [only f^n of x]

then $I.F = e^{\int f(x)\,dx}$

(IV) when $\frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) = f(y)$ then $I.F = e^{\int f(y)}$

Linear Diff. eq^n with Constant Coefficient

$$\frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + a_2 \frac{d^{n-2} y}{dx^{n-2}} + \cdots + a_n y = \phi(x)$$

$$\left[D^n + a_1 D^{n-1} + \cdots + a_n\right] y = \phi(x)$$

$\boxed{\text{General Sol}^n = C.F + P.I}$ where $D^n = \frac{d^n}{dx^n}$

finding Complementary function (C.F)

Step (1) let $(D^n + a_1 D^{n-1} + \cdots + a_n) = 0$

find the roots of above eq^n (called Auxiliary eq^n)

(1) when the roots are real and distinct

then C.f $y = c_1 e^{m_1 x} + c_2 e^{m_2 x} + c_3 e^{m_3 x} + \cdots$

where $m_1, m_2,$ are roots of A.E

(2) when the roots of A.E are repeated

let 2 roots are same $m_1 = m_2 = m$ (say)

then C.f

$$y = (c_1 x + c_2) e^{mx} + c_3 e^{m_3 x} + \cdots$$

let 3 roots are same $m_2 = m_2 = m_3 = m$ (say)

then C.f

$$y = (c_1 x^2 + c_2 x + c_3) e^{mx} + c_4 e^{m_4 x} + \cdots$$

and so on

if root m_1 is repeated for r times

$$y = (c_1 + c_2 x + c_3 x \ldots + c_r x^{r-1}) e^{mx} + c_{r+1} e^{m_{r+1} x} + \cdots$$

(3) when the roots of A.E are Complex then C.F

$$y = c_1 e^{(\alpha+i\beta)x} + c_2 e^{(\alpha-i\beta)x}$$

$$y = e^{\alpha x}\left[A\cos\beta x + B\sin\beta x\right]$$

if Complex roots are also repeated then C.F Can be found ~~by~~ as real roots.

Ex:- if Repeat two times

$$y = (c_1 + c_2 x)\, e^{(\alpha+i\beta)x} + (c_3 + c_4)\, e^{(\alpha-i\beta)x}$$

Finding particular Integral (P.I)

× ×

① $\dfrac{1}{(D-\alpha)} Q(x) = e^{\alpha x}\displaystyle\int Q(x)\, e^{-\alpha x}\, dx$

② $\dfrac{1}{f(D)} e^{ax} = \dfrac{e^{ax}}{f(a)}, \quad f(a) \neq 0$

③ $\dfrac{1}{f(D)} e^{ax} = \dfrac{1}{\phi(a)} \dfrac{x^r}{r!} e^{ax}, \quad \phi(a) \neq 0$

④ $\dfrac{1}{f(D^2)} \sin ax = \dfrac{1}{f(-a^2)} \sin ax, \quad \dfrac{1}{f(D^2)} \cos ax = \dfrac{1}{f(-a^2)} \cos ax$

⑤ $\dfrac{1}{D^2+a^2} \cos ax = \dfrac{x}{2a} \sin ax, \quad \dfrac{1}{D^2+a^2} \sin ax = -\dfrac{x}{2a} \cos ax$

⑥ $\dfrac{1}{f(D)} e^{ax}.V = e^{ax} \dfrac{1}{f(D+a)} V$, where V is a f^n of x

⑦ $\dfrac{1}{f(D)} x.V = x \dfrac{1}{f(D)} V + \left\{\dfrac{d}{dD} \dfrac{1}{f(D)}\right\} V$

See the examples of TIFR

Homogeneous Linear Differential equation

Gen form
$$x^n \frac{d^n y}{dx^n} + a_1 x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + a_2 x^{n-2} \frac{d^{n-2} y}{dx^{n-2}} + \dots + a_n y = Q(x)$$

where $a_1, a_2, a_3 \dots a_n$ are Constants

method of Solution

Put
$$Z = \log_e x \Rightarrow x = e^Z$$

and
$$x \frac{dy}{dx} = \frac{dy}{dZ} = Dy \quad \text{here } D = \frac{d}{dZ}$$

$$x^2 \frac{d^2 y}{dx^2} = D(D-1)y$$

$$x^3 \frac{d^3 y}{dx^3} = D(D-1)(D-2)y$$

$\vdots$

$$x^n \left(\frac{d^n}{dx^n}\right) = D(D-1)(D-2)\dots(D-n+1)$$

NOTE: ① don't forget to Substitute x with e^Z in Right hand side $[Q(x)]$

② Don't forget here $D = \frac{d}{dZ}$, not $\frac{d}{dy}$

Ex
$$x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} - y = x^2$$

Let $z = \log_e x$ i.e $x = e^Z$

$$\{[D(D-1)] + D - 1\} y = x^2 \quad \text{here } \frac{d}{dZ} = D$$

A.E $D^2 - D + D - 1 = 0$, $m = \pm 1$

$$C.F = c_1 e^Z + c_2 e^{-Z}$$

$$P.I = \frac{1}{D^2 - 1} e^{2Z} = \frac{1}{3} x^2$$

IIT JAM Solved Problems

2017 Consider the diff. eqn $y'' + 2y' + y = 0$

if $y(0) = 0$, $y'(0) = 1$ then value of $y(2) = ?$

Soln $\quad y'' + 2y' + y = 0$

Auxilary eqn $\quad (D^2 + 2D + 1)\,y = 0$

$(D^2 + 2D + 1) = 0 \Rightarrow (D+1)^2 = 0$

$D = -1, -1$

Roots are same $\quad$ Complenantory soln $= (A + Bx)e^{mx}$

$y = (A + Bx)\,e^{-x} \Rightarrow y' = Be^{-x} - Bxe^{-x} - Ae^{-x}$

put the given condition $\quad y(0) = 0$, $y'(0) = 1$

$0 = (A + B\times 0)\,e^{0} \Rightarrow A = 0,$ $\quad 1 = Be^{0} - B\times 0\,e^{0}$

$\boxed{B = 1}$

Soln $\quad y = x\,e^{-x}$

$$\boxed{y(2) = 2e^{-2} = 0.2706}$$

2015 $\quad$ eqn $\dfrac{dy}{dx} = \dfrac{y^2}{x}$ with boundary eqn $y(1) = 1$ out of the following the range of x in which y is real and finite, is

(A) $-\infty \le x \le -3$ $\quad$ (B) $-3 \le x \le 0$ $\quad$ (C) $0 \le x \le 3$ $\quad$ (D) $3 \le x \le \infty$

Soln $\quad \dfrac{dy}{dx} = y^2/x \Rightarrow \dfrac{1}{y^2}\,dy = \dfrac{1}{x}\,dx$

$-\dfrac{1}{y} = \log_e x + C$

Put $y(1) = 1$

$-\dfrac{1}{1} = 0 + C \Rightarrow \boxed{C = -1}$

$-\dfrac{1}{y} = \log_e x - 1$

$$\boxed{y = \frac{1}{1 - \log_e x}}$$

Now we see $\boxed{y = \frac{1}{1-\log x}}$

option (a) and (b) is dismissed because $\log(-x)$ is not defined.

option (c) is $0 \le x \le 3$ but we know if $\boxed{1-\log x = 0}$ then $y = \infty$

for $1-\log x = 0 \Rightarrow x = 2.71$

hence in Range $0 \le x \le 3$ for $x = 2.71$

y = infinite

option (D) Correct. You may check for $x \ge 3$ y will be Real and finite

2014

Find the sol^n $\frac{d^2y}{dx^2} + 5\frac{dy}{dx} = 0$, $y(0) = 2$, $y'(0) = 2$

Find x where $y = 0$

Sol^n A.E

$$(D^2 + 5D)\,y = 0$$

$$D(D+5) = 0, \Rightarrow D = 0, -5$$

C.F

$$y = Ae^0 + Be^{-5x}$$

$y(0) = 2 \Rightarrow 2 = A + Be^0 \Rightarrow \boxed{A+B = 2}$

$y'(0) = 2 \Rightarrow y' = -5Be^{-5x} \Rightarrow 2 = -5B \Rightarrow B = -2/5$

$A = 2 - (-2/5) = 12/5$

$$\boxed{y = \frac{12}{5} - \frac{2}{5}e^{-5x}}$$

when $y = 0$ then $\frac{2}{5}e^{-5x} = \frac{12}{5} \Rightarrow e^{-5x} = 6$

$\boxed{x = -\frac{1}{5}\ln 6} \Leftarrow -5x = \ln 6$

2005 Solve the diff eqⁿ $xy\frac{dy}{dx} = 3y^2 + x^2$ with initial Condition $y=2$, when $x=1$

Solⁿ $\frac{dy}{dx} = \frac{3y^2+x^2}{xy} \Rightarrow \frac{dy}{dx} = \frac{3y}{x} + \frac{x}{y}$

given eqⁿ is homogeneous D.E

put $y = vx \Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$

$$v + x\frac{dv}{dx} = 3v + \frac{1}{v}$$

$$x\frac{dv}{dx} = 2v + \frac{1}{v} \Rightarrow x\frac{dv}{dx} = \frac{2v^2+1}{v}$$

$$\int \frac{v}{2v^2+1}\,dv = \int \frac{1}{x}\,dx$$

[Now Let (in L.H.S) $2v^2+1 = t$

$4v\frac{dv}{dt} = 1 \Rightarrow v\,dv = \frac{1}{4}dt$

$$\frac{1}{4}\int \frac{1}{t}\,dt = \int \frac{1}{x}\,dx$$

$$\frac{1}{4}\log_e t = \log x + \log C$$

$$\frac{1}{4}\log_e(2v^2+1) = \log cx$$

$$\log_e\left(2\frac{y^2}{x^2}+1\right) = 4\log cx$$

Since $M\log N = \log N^M$

$$\boxed{\frac{2y^2}{x^2}+1 = c^4x^4}$$

$x=1$ then $y=2$

$$\frac{8}{1}+1 = c^4 1^4 \Rightarrow \boxed{c^4 = 9}$$

$$\boxed{C = \pm\sqrt{3}}$$

TIFR-2017 write down $x(t)$, where $x(t)$ is the sol^n of the following differential eq^n

$$\left(\frac{d}{dt}+2\right)\left(\frac{d}{dt}+1\right)x = 1$$

with boundary Condition $\left.\frac{dx}{dt}\right|_{t=0} = 0 \quad x|_{t\to\infty} = -1/2$

Sol^n Complementary function

$$[(D+2)(D+1)]x = 0 \qquad \text{where } D = \frac{d}{dt}$$

$$(D^2+3D+2) = 0 \Rightarrow D = -2, -1 \text{ (roots)}$$

C.F.

$$x = c_1 e^{-2t} + c_2 e^{-t}$$

applying Boundary Condition

$c_1 = 1/2, \; c_2 = -1$

Particular integral

$$P.I = \frac{1}{(D+2)(D+1)} * (1) = \frac{1}{(D^2+3D+2)} \times 1$$

$$= \frac{1}{2\left(1+\left(\frac{3D+D^2}{2}\right)\right)} \times 1 = \frac{1}{2}\left[1+\left(\frac{3D+D^2}{2}\right)\right]^{-1} \times 1$$

[using Binomial expansion]

$$= \frac{1}{2}\left[1 - \frac{3D+D^2}{2} + \frac{(3D+D^2)^2}{4} + \dots\right] \times 1$$

$P.I = \frac{1}{2}$ Since for $D, D^2, \dots$ derivatives of 1 will be zero

General Sol^n = C.F + P.I

$$\boxed{x = \tfrac{1}{2}e^{-2t} - e^{-t} + \tfrac{1}{2}}$$

TIFR-2013

$$\frac{d^2y}{dx^2} - \frac{2dy}{dx} + y = 0$$

Sol^n C.F = ? A.E $\Rightarrow$ $(D^2-2D+1)\,y = 0$

$(D^2-2D+1) = 0 \Rightarrow D = 1, 1$

$(D-1)^2 = 0$

roots are similar

C.F $y = (C_1 + C_2x)\,e^{mx}$

$$\boxed{y = (C_1+C_2x)\,e^{x}}$$

JNU general Sol^n of

$$x^2\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 2y = 0$$

interms of two arbitrary Constant

Sol^n $z = \log_e x \Rightarrow x = e^z$

$$[D(D-1) - 2D + 2]\,y = 0$$

A.E $D^2 - D - 2D + 2 = 0 \Rightarrow D^2 - 3D + 2 = 0$

$D^2 - D - 2D + 2 = 0 \Rightarrow D(D-1) - 2(D-1) = 0$

$(D-1)(D-2) = 0$, $D = 1, 2$ (roots)

C.F $= C_1e^z + C_2e^{2z} = C_1x + C_2x^2$

P.I = 0 Since $\phi(x) = 0$

Sol^n $\boxed{y = C_1x + C_2x^2}$

Gate 2014

$$\frac{d^2y}{dt^2} - y = 0 \text{, boundary Condition } \begin{array}{l} y(0) = 1 \\ y(\infty) = 0 \end{array}$$

Sol^n A.E $(D^2 - 1)y = 0$ here $D = \frac{d}{dt}$

$D = \pm 1$ (roots)

[using formula for real & distinct roots]

C.F $\boxed{y = Ae^{t} + Be^{-t}}$

at $t = 0$, $y = A + B = 1$, $B = 1$

at $t = \infty$ $y = Ae^{\infty} + \frac{B}{e^{\infty}} = 0 \Rightarrow A = 0$

$$\boxed{y = e^{-t}}$$

Gate 2016

$\frac{dy}{dx} = xy$ if $y = 2$ at $x = 0$ then Value of y at $x = 2$

Sol^n

Variable separation method

$$\int \frac{1}{y} dy = \int x \, dx$$

$$\ln y = \frac{x^2}{2} + C$$

apply boundary Condition at $x = 0$, $y = 2$

$\ln 2 = C$

$$\ln y - \ln 2 = x^2/2$$

y at $x = 2$

$$\ln y = 4/2 + \ln 2 = 2 + 0.693$$

$$y = e^{2.693} = 14.78 = 2e^2$$

Net-June 2015

$$\frac{dy}{dx} = x^2 - y$$ with initial Condition $y=2$ at $x=0$

$$\frac{dy}{dx} + y = x^2$$ given eqn is in form L.D.E

$p(x) = 1$, $Q(x) = x^2$

$$I.F. = e^{\int 1\,dx} = e^{x}$$

$$y.e^{x} = \int e^{x} x^2\,dx$$

$$y e^{x} = x^2 e^{x} - \int 2x e^{x}\,dx + C$$

$$= x^2 e^{x} - 2(x e^{x} - e^{x}) + C$$

$$y e^{x} = x^2 e^{x} - 2x e^{x} - e^{x} + C$$

at $x=0$, $y=2$ $\Rightarrow$ $C=3$

$$\boxed{y = x^2 - 2x - 1 + 3e^{-x}}$$

NET 2015 Dec.

$$\frac{dx}{dt} = 2\sqrt{1-x^2}$$ with initial Condition $x=0$ at $t=0$

$$\int \frac{dx}{\sqrt{1-x^2}} = 2\int dt \Rightarrow \boxed{\sin^{-1} x = 2t + C}$$

Since $\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x + C$

applying boundary Condition $\sin^{-1}(0) = 2\times 0 + C$

$$\boxed{C=0}$$

$$\boxed{\sin^{-1} x = 2t} \Rightarrow \boxed{x = \sin 2t}$$

Derivatives

1) $\frac{d}{dx} x^n = n x^{n-1}$

2) $\frac{d}{dx} \sin x = \cos x$

3) $\frac{d}{dx} \cos x = -\sin x$

4) $\frac{d}{dx} \tan x = \sec^2 x$

5) $\frac{d}{dx} \cot x = -\operatorname{cosec}^2 x$

6) $\frac{d}{dx} \sec x = \sec x \tan x$

7) $\frac{d}{dx} \operatorname{cosec} x = -\operatorname{cosec} x \cot x$

8) $\frac{d}{dx} \sin^{-1} x = \frac{1}{\sqrt{1-x^2}}$

9) $\frac{d}{dx} \cos^{-1} x = \frac{-1}{\sqrt{1-x^2}}$

10) $\frac{d}{dx} \tan^{-1} x = \frac{1}{1+x^2}$

11) $\frac{d}{dx} \cot^{-1} x = \frac{-1}{1+x^2}$

12) $\frac{d}{dx} \sec^{-1} x = \frac{1}{x\sqrt{x^2-1}}$

13) $\frac{d}{dx} \operatorname{cosec}^{-1} x = \frac{-1}{x\sqrt{x^2-1}}$

14) $\frac{d}{dx} e^x = e^x$

15) $\frac{d}{dx} a^x = \log a, a^x$

16) $\frac{d}{dx} \log|x| = \frac{1}{x}$

Integrals

$\int x^n dx = \frac{x^{n+1}}{n+1} + c$

$\int \cos x \, dx = \sin x + C$

$\int \sin x \, dx = -\cos x + C$

$\int \sec^2 x \, dx = \tan x + c$

$\int \operatorname{cosec}^2 x \, dx = -\cot x + c$

$\int \sec x \tan x \, dx = \sec x + C$

$\int \operatorname{cosec} x \cot x \, dx = -\operatorname{cosec} x +$

$\int \frac{1}{\sqrt{1-x^2}} dx = \sin^{-1} x + C$

$\int \frac{-1}{\sqrt{1-x^2}} dx = \cos^{-1} x + C$

$\int \frac{dx}{1+x^2} = \tan^{-1} x + C$

$\int \frac{-1 dx}{1+x^2} = + \cot^{-1} x +$

$\int \frac{1}{x\sqrt{x^2-1}} dx = \sec^{-1} x +$

$\int \frac{-1}{x\sqrt{x^2-1}} dx = + \operatorname{cosec}^{-1} x$

$\int e^x dx = e^x + C$

$\int a^x dx = \frac{a^x}{\log a} + C$

$\int \frac{1}{x} dx = \log|x| + C$

Imp. Integral formula

(i) $\int \tan x \, dx = \log|\sec x| + C$

(ii) $\int \cot x \, dx = \log|\sin x| + C$

(iii) $\int \sec x \, dx = \log|\sec x + \tan x| + C$

(iv) $\int \operatorname{cosec} x \, dx = \log|\operatorname{cosec} x - \cot x| + C$

(v) $\int \frac{dx}{x^2 - a^2} = \frac{1}{2a} \log\left|\frac{x-a}{x+a}\right| + C$

(vi) $\int \frac{dx}{a^2 - x^2} = \frac{1}{2a} \log\left|\frac{a+x}{a-x}\right| + C$

(vii) $\int \frac{dx}{x^2 + a^2} = \frac{1}{a} \tan^{-1}\frac{x}{a} + C$

(viii) $\int \frac{dx}{\sqrt{x^2 - a^2}} = \log\left|x + \sqrt{x^2 - a^2}\right| + C$

(ix) $\int \frac{dx}{\sqrt{x^2 + a^2}} = \log\left|x + \sqrt{x^2 + a^2}\right| + C$

(x) $\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\frac{x}{a} + C$

(xi) $\int \sqrt{x^2 - a^2} \, dx = \frac{x}{2}\sqrt{x^2 - a^2} - \frac{a^2}{2} \log\left|x + \sqrt{x^2 - a^2}\right| + C$

(xii) $\int \sqrt{x^2 + a^2} \, dx = \frac{x}{2}\sqrt{x^2 + a^2} + \frac{a^2}{2} \log\left|x + \sqrt{x^2 + a^2}\right| + C$

(xiii) $\int \sqrt{a^2 - x^2} \, dx = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2} \sin^{-1}\frac{x}{a} + C$

~~(xiv)~~